1 Introduction

In modern statistics and high-dimensional data analysis, the study of large random matrices has become increasingly important. When analyzing datasets with many variables relative to the number of observations, classical statistical methods often fail, and new theoretical frameworks are needed. Random matrix theory provides powerful tools for understanding the behavior of such high-dimensional systems.

A fundamental object in multivariate statistics is the *data matrix*, which organizes n observations of p variables into a structured $p \times n$ matrix $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, where each column $\mathbf{x}_i$ corresponds to a single observation and each row represents a variable. This representation facilitates computational efficiency and provides a natural framework for multivariate analysis techniques.

As the dimensions of the data matrix grow large, the empirical properties of derived quantities, such as the sample covariance matrix

$$\mathbf{S}_n = \frac{1}{n}\mathbf{X}\mathbf{X}^*,$$

become increasingly important. Here, $\mathbf{X}^*$ denotes the conjugate transpose of $\mathbf{X}$.

A cornerstone result in random matrix theory is the *Marčenko-Pastur law* [2], which describes the limiting spectral distribution of sample covariance matrices. Specifically, if the entries of $\mathbf{X}$ are independent and identically distributed (i.i.d.) with mean zero and variance one, and if the ratio $p/n \to y > 0$ as $n \to \infty$, then the empirical spectral distribution of $\mathbf{S}_n$ converges almost surely to a nonrandom limit whose density is given by the Marčenko-Pastur distribution.

1.1 Motivation and Related Work

The classical Marčenko-Pastur law has been extended in numerous directions. Important generalizations include the work of Silverstein [3] on matrices with non-i.i.d. entries, Bai and Silverstein [4] on central limit theorems for linear spectral statistics, and more recent developments in free probability theory [5].

In many practical applications, one encounters not just sample covariance matrices themselves, but products or linear combinations of such matrices with other structured matrices. For instance, in multivariate analysis, Fisher matrices of the form $\mathbf{F}_n = \mathbf{S}_n\mathbf{S}_n^{-1}$ arise naturally [6]. Similarly, in signal processing and wireless communications, products of random matrices with deterministic sequences appear in the analysis of channel capacity and detection algorithms.

1.2 Main Contributions

In this paper, we study the limiting spectral behavior of matrices of the form

$$\mathbf{B}_n = \mathbf{S}_n\mathbf{T}_n,$$

where $\mathbf{S}_n$ is the sample covariance matrix and $\{\mathbf{T}_n\}$ is a sequence of nonnegative definite Hermitian matrices.

We build upon existing convergence results [3, 4] that establish the almost sure convergence of the empirical spectral distribution of $\mathbf{B}_n$ and provide implicit characterizations through Stieltjes transforms. Our main contribution is:

1. We derive explicit bounds for the support of the limiting distribution when the limiting measure H has a two-point support structure of the form $H = \beta\delta_a + (1-\beta)\delta_1$.
2. We provide a complete proof technique using contradiction arguments and the Intermediate Value Theorem to establish these bounds.
3. We demonstrate the computational applicability through the alternative formulation involving the related matrix $\underline{\mathbf{B}}_n$.

2 Preliminaries and Notation

Let $\mathbf{M}$ be a $p\times p$ Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_p$. The *empirical spectral distribution* (ESD) of $\mathbf{M}$ is defined as

$$F^{\mathbf{M}} = \frac{1}{p}\sum_{i=1}^{p} \delta_{\lambda_i},$$

where δ_{λ_i} is the Dirac delta measure at λ_i. Equivalently, the cumulative distribution function is given by

$$\mu(x) = F^{\mathbf{M}}[0, x] = \frac{1}{p}\sum_{i=1}^{p} \mathbf{1}_{\lambda_i \le x},$$

where $\mathbf{1}_{\lambda_i \le x}$ is the indicator function.

The *Stieltjes transform* of a probability measure μ on $\mathbb{R}$ is defined as

$$s_\mu(z) = \int_{\mathbb{R}} \frac{1}{x-z}\, d\mu(x), \quad z \in \mathbb{C}^+,$$

where $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ is the upper half-plane.

3 Background Theory

3.1 General Convergence Result

We begin by recalling a fundamental result from random matrix theory that extends the classical Marčenko-Pastur law to products of sample covariance matrices with deterministic sequences. This result, due to Silverstein and Bai, forms the foundation for our analysis.

Theorem 3.1 (Silverstein-Bai Convergence Theorem, [3, 4]). *Let $\mathbf{S}_n$ be the sample covariance matrix defined by $\mathbf{S}_n = \frac{1}{n}\mathbf{X}\mathbf{X}^*$, where $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is a $p \times n$ data matrix. Let $\{\mathbf{T}_n\}$ be a sequence of nonnegative definite Hermitian matrices of size $p \times p$. Define*

$$\mathbf{B}_n = \mathbf{S}_n\mathbf{T}_n. \tag{1}$$

Assume the following conditions hold:

1. *The entries* (x_{jk}) *of the data matrix are i.i.d. with mean zero and variance 1.*
2. *The dimension-to-sample ratio satisfies* $\frac{p}{n} \to y > 0$ *as* $n \to \infty$.
3. *The sequence* $\{\mathbf{T}_n\}$ *is either deterministic or independent of* $\mathbf{S}_n$.
4. *Almost surely, the empirical spectral distribution* $H_n = F^{\mathbf{T}_n}$ *of* $\mathbf{T}_n$ *converges weakly to a nonrandom probability measure* H.

Then almost surely, $F^{\mathbf{B}_n}$ *converges weakly to a nonrandom probability measure* $F_{y,H}$. *Moreover, the Stieltjes transform* $s(z)$ *of* $F_{y,H}$ *satisfies the implicit equation*

$$s(z) = \int \frac{1}{t(1-y-yzs(z))-z}\,dH(t), \quad z \in \mathbb{C}^+. \tag{2}$$

Remark 3.2. *The equation* (2) *uniquely determines the Stieltjes transform* $s(z)$ *in the upper half-plane, and hence uniquely determines the limiting distribution* $F_{y,H}$.

3.2 Stieltjes Transform Inversion

The following classical result from complex analysis provides the standard method for recovering probability measures from their Stieltjes transforms. This inversion formula is a cornerstone tool in random matrix theory.

Proposition 3.3 (Classical Stieltjes Inversion Formula, [1])**.** *Let* μ *be a probability measure on* $\mathbb{R}$ *with Stieltjes transform* $s_\mu(z)$. *Then for all continuous and compactly supported functions* $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{R}} \varphi(x)\,\mu(dx) = \lim_{v \downarrow 0} \frac{1}{\pi} \int_{\mathbb{R}} \varphi(x)\,\Im(s_\mu(x+iv))\,dx.$$

In particular, for continuity points $a < b$ *of* μ,

$$\mu([a,b]) = \lim_{v \downarrow 0} \frac{1}{\pi} \int_a^b \Im(s_\mu(x+iv))\,dx.$$

4 Applications of Background Theory

The convergence result in Theorem 3.1 can be illustrated through several classical examples from random matrix theory and multivariate statistics.

4.1 Classical Marčenko-Pastur Law

Example 4.1 (Identity Matrix Case)**.** *If* $\mathbf{T}_n = \mathbf{I}_p$ *(the* $p \times p$ *identity matrix), then* $\mathbf{B}_n = \mathbf{S}_n$. *In this case,* $H = \delta_1$ *(the Dirac measure at 1), and Theorem 3.1 reduces to the classical Marčenko-Pastur law. The limiting distribution has support on* $[(1-\sqrt{y})^2, (1+\sqrt{y})^2]$ *when* $y \le 1$, *and has an additional point mass at zero when* $y > 1$.

4.2 Fisher Matrices

Example 4.2 (Fisher Matrix)**.** *In multivariate analysis, the Fisher matrix is defined as* $\mathbf{F}_n = \mathbf{S}_n\mathbf{S}_n^{-1}$*, where* $\mathbf{S}_n$ *and* $\mathbf{S}_n^{-1}$ *are independent sample covariance matrices. This can be written in the form* $\mathbf{F}_n = \mathbf{S}_n\mathbf{T}_n$ *with* $\mathbf{T}_n = \mathbf{S}_n^{-1}$*.*

Under the conditions of Theorem 3.1, and assuming that the spectral distribution of $\mathbf{T}_n$ *converges appropriately, the asymptotic behavior of* $\mathbf{F}_n$ *can be analyzed using this framework. The limiting spectral distribution of Fisher matrices has been studied extensively by Wachter [6] and later by Silverstein and Bai [3].*

4.3 Cross Data Matrices

Example 4.3 (Product of Independent Sample Covariance Matrices)**.** *Consider the case where* $\mathbf{T}_n$ *is an independent copy of the sample covariance matrix, denoted* $\mathbf{S}_{2,n}$*. We define*

$$\mathbf{B}_n = \mathbf{S}_{1,n}\mathbf{S}_{2,n},$$

where $\mathbf{S}_{1,n}$ *and* $\mathbf{S}_{2,n}$ *are independent sample covariance matrices. Such products arise in the analysis of cross-correlation matrices in signal processing.*

Under the framework of Theorem 3.1, assuming appropriate convergence of the spectral distributions of both matrices, one can analyze the asymptotic spectral properties of $\mathbf{B}_n$*. The limiting spectral distribution is characterized by the implicit equation* (2) *with* H *being the Marčenko-Pastur distribution.*

5 Main Contribution: Explicit Support Bounds

Building upon the convergence result in Theorem 3.1, we now present our main original contribution: explicit bounds for the support of the limiting distribution in a special case. While Theorem 3.1 provides the existence of the limiting distribution and its implicit characterization via the Stieltjes transform, it does not give explicit information about the support of this distribution. Our work fills this gap for an important special case.

Remark 5.1. *It is important to emphasize that Theorem 3.1 and Proposition 3.3 are existing results from the literature that we utilize as tools. Our original contribution is Theorem 5.2 below, which provides explicit support bounds that were not previously available.*

5.1 Support Bounds for Two-Point Measures

We consider a specific case where the limiting measure H has a simple two-point structure, allowing us to derive explicit and computable bounds for the support of the limiting distribution.

Theorem 5.2 (Support Bounds for Two-Point Measures - Main Result)**.** *Consider the setting of Theorem 3.1, and suppose the measure* H *is specified as*

$$H = \beta\delta_a + (1-\beta)\delta_1,$$

where $a \geq 1$, $y \geq 1$, and $0 \leq \beta \leq 1$. Then the support of $F_{y,H}$ is contained in the interval

$$\left[\max_{k \in (0,\infty)} \psi(k), \min_{t \in (-\frac{1}{a},0)} \psi(t) \right], \tag{3}$$

where

$$\psi(u) = \frac{y\beta(a-1)u + (au+1)((y-1)u-1)}{(au+1)(u^2+u)}.$$

5.2 Alternative Formulation

For computational purposes, it is often useful to work with an alternative formulation involving the matrix $\underline{\mathbf{B}}_n = \frac{1}{n}\mathbf{X}^*\mathbf{T}_n\mathbf{X}$ of size $n \times n$. The matrices $\mathbf{B}_n$ and $\underline{\mathbf{B}}_n$ share the same nonzero eigenvalues, so their ESDs satisfy

$$nF^{\underline{\mathbf{B}}_n} - pF^{\mathbf{B}_n} = (n-p)\delta_0. \tag{4}$$

When $p/n \to y > 0$, this gives the relation

$$\underline{F}_{y,H} - yF_{y,H} = (1-y)\delta_0, \tag{5}$$

where $\underline{F}_{y,H}$ is the limiting distribution of $\underline{\mathbf{B}}_n$.

The corresponding Stieltjes transforms $\underline{s}(z)$ and $s(z)$ are related by

$$\underline{s}(z) = -\frac{1-y}{z} + ys(z). \tag{6}$$

Substituting this relation into the equation for $s(z)$, we obtain the alternative formulation

$$\underline{s}(z) = -\left(z - y\int \frac{t}{1+t\underline{s}(z)}\, dH(t)\right)^{-1}. \tag{7}$$

Solving for z yields

$$z = -\frac{1}{\underline{s}(z)} + y\int \frac{t}{1+t\underline{s}(z)}\, dH(t). \tag{8}$$

6 Proof of the Main Result

6.1 Proof of Theorem 5.2 (Support Bounds)

Proof. We utilize the alternative formulation (8) to determine the support of the limiting distribution. For the two-point measure $H = \beta\delta_a + (1-\beta)\delta_1$, equation (8) becomes

$$\begin{aligned} z &= -\frac{1}{\underline{s}} + y\int \frac{t}{1+t\underline{s}}\,dH(t) \\ &= -\frac{1}{\underline{s}} + y\left(\frac{\beta a}{1+a\underline{s}} + \frac{1-\beta}{1+\underline{s}}\right) \\ &= -\frac{1}{\underline{s}} + y\frac{\beta a(1+\underline{s}) + (1-\beta)(1+a\underline{s})}{(1+a\underline{s})(1+\underline{s})} \\ &= -\frac{1}{\underline{s}} + y\frac{\beta a + 1 - \beta + a\underline{s}}{(1+a\underline{s})(1+\underline{s})}. \end{aligned} \tag{9}$$

Multiplying both sides by $\underline{s}(1+a\underline{s})(1+\underline{s})$ and rearranging, we obtain

$$(a\underline{s}+1)\left(z\underline{s}^2 + (z-y+1)\underline{s} + 1\right) = y\beta(a-1)\underline{s}. \tag{10}$$

This can be rewritten as

$$z\underline{s}^2 + (z-y+1)\underline{s} + 1 = \frac{y\beta(a-1)\underline{s}}{a\underline{s}+1}. \tag{11}$$

To determine the extreme values of z for which this equation has a unique real solution, we define

$$f(\underline{s}) = z\underline{s}^2 + (z-y+1)\underline{s} + 1, \tag{12}$$

$$g(\underline{s}) = \frac{y\beta(a-1)\underline{s}}{a\underline{s}+1}. \tag{13}$$

The function $f(\underline{s})$ is quadratic, while $g(\underline{s})$ is a rational function with a vertical asymptote at $\underline{s} = -\frac{1}{a}$ and horizontal asymptote at $y\beta\frac{a-1}{a}$.

For the upper bound $z_{\max}$, we consider the interval $(-\frac{1}{a}, 0)$. We argue by contradiction as follows. Suppose there exists $t \in (-\frac{1}{a}, 0)$ (with $0 > t > -\frac{1}{a} \geq -1$) such that

$$z_{\max} > \frac{y\beta(a-1)t + (at+1)((y-1)t-1)}{(at+1)(t^2+t)}. \tag{14}$$

Then, when $z = z_{\max}$, we compute

$$\begin{aligned}
f(t) - g(t) &= z_{\max}t^2 + (z_{\max} - y + 1)t + 1 - \frac{y\beta(a-1)t}{at+1} \\
&= z_{\max}(t^2 + t) + (1-y)t + 1 - \frac{y\beta(a-1)t}{at+1} \\
&< \frac{y\beta(a-1)t + (at+1)((y-1)t-1)}{(at+1)(t^2+t)}(t^2+t) + (1-y)t + 1 - \frac{y\beta(a-1)t}{at+1} \\
&= \frac{y\beta(a-1)t + (at+1)((y-1)t-1)}{at+1} + (1-y)t + 1 - \frac{y\beta(a-1)t}{at+1} \\
&= \frac{y\beta(a-1)t}{at+1} + \frac{(at+1)((y-1)t-1)}{at+1} + (1-y)t + 1 - \frac{y\beta(a-1)t}{at+1} \\
&= \frac{(at+1)((y-1)t-1)}{at+1} + (1-y)t + 1 \\
&= (y-1)t - 1 + (1-y)t + 1 \\
&= 0.
\end{aligned}$$

Since

$$\begin{cases} f(0) - g(0) = 1 > 0, \\ \lim_{\underline{s} \to (-\frac{1}{a})^+} (f(\underline{s}) - g(\underline{s})) = +\infty, \\ \text{and there exists some } t \in (-\frac{1}{a}, 0) \text{ such that } f(t) - g(t) < 0, \end{cases} \tag{15}$$

the Intermediate Value Theorem guarantees the existence of at least two distinct real solutions $\theta, \phi \in (-\frac{1}{a}, 0)$ such that

$$f(\theta) = g(\theta) \quad \text{and} \quad f(\phi) = g(\phi). \tag{16}$$

This contradicts the uniqueness required for the definition of $z_{\max}$.

Since, for every $t \in (-\frac{1}{a}, 0)$, we have shown that

$$z_{\max} \leq \frac{y\beta(a-1)t + (at+1)((y-1)t-1)}{(at+1)(t^2+t)}, \tag{17}$$

taking the minimum over t in $(-\frac{1}{a}, 0)$ yields

$$z_{\max} \leq \min_{t \in (-\frac{1}{a}, 0)} \frac{y\beta(a-1)t + (at+1)((y-1)t-1)}{(at+1)(t^2+t)}. \tag{18}$$

For the lower bound $z_{\min}$, we consider the interval $(0, \infty)$. For all $k \in (0, \infty)$ (with $k > 0$), we argue by contradiction as follows. Suppose

$$z_{\min} < \frac{y\beta(a-1)k + (ak+1)((y-1)k-1)}{(ak+1)(k^2+k)}. \tag{19}$$

Then, when $z = z_{\min}$, we compute

$$\begin{aligned}
f(k) - g(k) &= z_{\min}k^2 + (z_{\min} - y + 1)k + 1 - \frac{y\beta(a-1)k}{ak+1} \\
&= z_{\min}(k^2 + k) + (1-y)k + 1 - \frac{y\beta(a-1)k}{ak+1} \\
&< \frac{y\beta(a-1)k + (ak+1)((y-1)k-1)}{(ak+1)(k^2+k)}(k^2+k) + (1-y)k + 1 - \frac{y\beta(a-1)k}{ak+1} \\
&= \frac{y\beta(a-1)k + (ak+1)((y-1)k-1)}{ak+1} + (1-y)k + 1 - \frac{y\beta(a-1)k}{ak+1} \\
&= \frac{y\beta(a-1)k}{ak+1} + \frac{(ak+1)((y-1)k-1)}{ak+1} + (1-y)k + 1 - \frac{y\beta(a-1)k}{ak+1} \\
&= \frac{(ak+1)((y-1)k-1)}{ak+1} + (1-y)k + 1 \\
&= (y-1)k - 1 + (1-y)k + 1 \\
&= 0.
\end{aligned}$$

Since

$$\begin{cases} f(0) - g(0) = 1 > 0, \\ \lim_{\underline{s}\to\infty}(f(\underline{s}) - g(\underline{s})) = +\infty, \\ \text{and there exists some } k \in (0,\infty) \text{ such that } f(k) - g(k) < 0, \end{cases} \tag{20}$$

the Intermediate Value Theorem guarantees the existence of at least two distinct real solutions $\theta, \phi \in (0, \infty)$ such that

$$f(\theta) = g(\theta) \quad \text{and} \quad f(\phi) = g(\phi). \tag{21}$$

This contradicts the uniqueness required for the definition of $z_{\min}$.

Since, for every $k \in (0, \infty)$, we have shown that

$$z_{\min} \geq \frac{y\beta(a-1)k + (ak+1)((y-1)k-1)}{(ak+1)(k^2+k)}, \tag{22}$$

taking the maximum over k in $(0, \infty)$ yields

$$z_{\min} \geq \max_{k\in(0,\infty)} \frac{y\beta(a-1)k + (ak+1)((y-1)k-1)}{(ak+1)(k^2+k)}. \tag{23}$$

The support of $F_{y,H}$ is therefore contained in the interval

$$\left[\max_{k\in(0,\infty)} \psi(k), \min_{t\in(-\frac{1}{a},0)} \psi(t)\right], \tag{24}$$

where $\psi(u) = \frac{y\beta(a-1)u+(au+1)((y-1)u-1)}{(au+1)(u^2+u)}$. □

7 Conclusion

Building upon existing convergence results for the limiting spectral distributions of products of sample covariance matrices with deterministic sequences, we have derived explicit bounds for the support of the limiting distribution in the important special case of two-point limiting measures. Our main theorem provides concrete upper and lower bounds that can be computed through optimization problems involving elementary functions.

The proof technique, based on contradiction arguments and the Intermediate Value Theorem, demonstrates how geometric properties of quadratic and rational functions can be leveraged to obtain precise spectral bounds. The alternative formulation using the related matrix $\underline{\mathbf{B}}_n$ proves particularly useful for the analysis.

These results have potential applications in various areas of statistics, signal processing, and wireless communications where products of random matrices with structured deterministic sequences arise naturally. The explicit nature of our bounds makes them particularly valuable for computational applications and numerical verification.

Future research directions include extending the bounds to more general support structures for the limiting measure H, investigating the sharpness of the derived bounds, and developing efficient numerical algorithms for computing the extremal values in the optimization problems that define the support boundaries.

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References

[1] N. I. Akhiezer. *The Classical Moment Problem and Some Related Questions in Analysis.* Oliver and Boyd, Edinburgh, 1965.

[2] V. A. Marčenko and L. A. Pastur. Distribution of eigenvalues for some sets of random matrices. *Mathematics of the USSR-Sbornik*, 1(4):457–483, 1967.

[3] J. W. Silverstein. Strong convergence of the empirical distribution of eigenvalues of large dimensional random matrices. *Journal of Multivariate Analysis*, 55(2):331–339, 1995.

[4] Z. Bai and J. W. Silverstein. CLT for linear spectral statistics of large-dimensional sample covariance matrices. *The Annals of Probability*, 32(1A):553–605, 2004.

[5] D. Voiculescu. Limit laws for random matrices and free products. *Inventiones Mathematicae*, 104(1):201–220, 1991.

[6] K. W. Wachter. The limiting empirical measure of multiple discriminant ratios. *The Annals of Statistics*, 6(5):1–22, 1978.